DUAL VECTOR SPACES, TRACES, AND CYCLICITY OF TRACE

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1. Introduction

This paper's purpose is to describe the dual of finite dimensional vector spaces V over an arbitrary field $\mathbb{F}$. Specifically, the aim is to show that V is finite dimensional, then construct a basis for it, and use this basis to define the trace of a linear map and show that it is independent of the basis chosen. The paper will conclude with a discussion of the cyclic property of traces.

2. Dual space of a finite dimensional vector space

Definition 2.1. Let V be a finite vector space over $\mathbb{F}$. The **dual space** of V is the vector space of all linear functionals on V such that $V^* = \mathcal{L}(V, \mathbb{F})$

Definition 2.2. If $e_1, \ldots, e_n$ is a basis of V, then the dual basis of $e_1, \ldots, e_n$ is the list $f_1, \ldots, f_n$ of elements of V^*, where each f_j is the linear functional on V such that $f_i(e_j) = 1$ if $j = i$ and $f_i(e_j) = 0$ otherwise.

Next, we'll see that $f_1, \ldots, f_n$ is also a basis for V^*. To see that $f_1, \ldots, f_n$ is linearly independent in V^*, we'll use the fact that only the i element is preserved when the linear map operates on the basis vector e, guaranteeing that all but the i terms will go to zero. And because $f_i(e_i) = 1$ for the i term, we are left with the scalar a_i. We'll then use our knowledge of dimensions of vector spaces and their relations to bases to show that $f_1, \ldots, f_n$ is a basis for V^*

Proposition 2.3. *The dual basis is a basis of the dual space* $V^* = \mathcal{L}(V, \mathbb{F})$

Proof. Suppose V is a finite dimensional vector space over $\mathbb{F}$ and $e_1, \ldots, e_n$ is a basis in V. Let $a_1, \ldots, a_n \in \mathbb{F}$ and define $f \in V^*$ as

$$f = a_1 f_1 + \ldots + a_n f_n$$

and suppose

$$0 = a_1 f_1 + \ldots + a_n f_n$$

By definition 2.2, $f_i(e_i) = (a_1 f_1 + \cdots + a_n f_n)(e_i) = a_i f_i(e_i) = a_i$ for $i = 1, \ldots n$. Setting this to zero as above clearly yields $a_1 = \ldots = a_n = 0$, thus showing $f_1, \ldots, f_n$ is linearly independent in V^*. Also note that the dimension of the dual space V^* is equal to the dimension of the original vector space V, i.e. that $dim(V^*) = dim(V) = n$. Then $f_1, \ldots, f_n$ is a basis in V^* because every linearly independent list of vectors in V^* with length $dim(V^*)$ is a basis of V^*. □

3. Traces

Proposition 3.1. *The trace of the the linear map $A \in \mathcal{L}(V, V)$, defined as the trace of the matrix of A with respect to some basis $e_1, \ldots, e_n$, is independent of the chosen basis*

Proof. Suppose V is a finite dimensional vector space over $\mathbb{F}$. Let $e_1, \ldots, e_n$ and $e'_1, \ldots, e'_n$ be bases for V and let $f_1, \ldots, f_n$ and $f'_1, \ldots, f'_n$ be their corresponding dual bases. Define Trace A as

$$\operatorname{Tr} A = \sum_{j=1}^{n} f_j(Ae_j)$$

and note that $f_j(Ae_j)$ is the jj entry on the matrix of A and that the trace is the sum of the diagonal entries of the matrix associated with A and the basis $e_1, \ldots, e_n$. Let

$$\begin{aligned} T &= \mathcal{M}(I) \\ M &= \mathcal{M}(Ae_j) \\ M' &= \mathcal{M}(Ae'_j) \end{aligned}$$

where I is the identity matrix with respect to bases $e_1, \ldots, e_n$ and $e'_1, \ldots, e'_n$. If we let $M = T^{-1}M'T$ by the change of basis formula, then

$$\begin{aligned} \sum_{j=1}^{n} f_j(Ae_j) &= \operatorname{Tr} A \\ &= \operatorname{Tr} M \\ &= \operatorname{Tr} T^{-1}M'T \\ &= \sum_{i,j,k} (T^{-1})_{ij} M'_{jk} T_{ki} \\ &= \sum_{i=1}^{n} \sum_{j,k} M'_{jk}(T_{ki}(T^{-1})_{ij})) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} M'_{jk}(TT^{-1})_{kj} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \sum_{k=1}^{n} M'_{jk} I_{kj} \\
&= \sum_{j=1}^{n} M'_{jj} \\
&= \operatorname{Tr} M' \\
&= \sum_{j=1}^{n} f'_j(Ae'_j)
\end{aligned}
$$

As desired. Hence, there is no need to specify a basis when discussing the trace of a linear operator or it's associated matrix. □

Example 3.2. *Suppose $A \in \mathcal{L}(V, V)$ and let the list e_1, e_2, e_3 be a basis in V with f_1, f_2, f_3 as its corresponding dual basis. Show that $\operatorname{Tr} A$ is equal to the sum of the diagonal entries on the matrix of A.*

Let $Ae_j = \sum a_{jk} e_j$ be the matrix associated with A and the basis e_1, e_2, e_3. Then

$$
\begin{aligned}
\operatorname{Tr} A &= \sum_{j=1}^{3} f_j(Ae_j) \\
&= \begin{bmatrix} f_1, f_2, f_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \\
&= \begin{bmatrix} f_1, f_2, f_3 \end{bmatrix} \begin{bmatrix} a_{11}e_1 + a_{12}e_2 + a_{13}e_3 \\ a_{21}e_1 + a_{22}e_2 + a_{23}e_3 \\ a_{31}e_1 + a_{32}e_2 + a_{33}e_3 \end{bmatrix} \\
&= f_1(a_{11}e_1 + a_{12}e_2 + a_{13}e_3) + f_2(a_{21}e_1 + a_{22}e_2 + a_{23}e_3) + f_3(a_{31}e_1 + a_{32}e_2 + a_{33}e_3) \\
&= f_1(a_{11}e_1) + 0 + 0 + 0 + f_2(a_{22}e_2) + 0 + 0 + 0 + f_3(a_{33}e_3) \\
&= a_{11}f_1e_1 + a_{22}f_2e_2 + a_{33}f_3e_3 \\
&= a_{11} + a_{22} + a_{33}
\end{aligned}
$$

as desired.

The next result follows from 3.1 and greatly simplifies our definition of trace. Because all of the non-jj entries on the matrix associated with A go to zero, the next definition says that, given any square matrix, we can ignore all but the jj entries and simply take the sum of the main diagonal without referring to any basis.

Definition 3.3. Given any square matrix A, the **trace** of A is the sum of the entries on the main diagonal. i.e.

$$\operatorname{Tr} A = \sum_{j=1}^{n} a_{jj} = a_{11} + a_{22} + \ldots + a_{nn}$$

where a_{jj} denotes the entry on the jth row and jth column of A.

Example 3.4. *Given an $n \times n$ identity matrix I, show that $\operatorname{Tr} I = n$ when $n = 3$. I.e., show that the diagonals of the 3×3 identity matrix add up to n.*

$$\begin{aligned} \operatorname{Tr} I &= \sum_{j=1}^{3} i_{jj} \\ &= 1 + 1 + 1 \\ &= 3 \end{aligned}$$

which is equal to n as desired.

Example 3.5. *Let $A = \begin{bmatrix} 6 & 13 & 2 \\ 2 & 1 & 0 \\ 4 & 9 & 3 \end{bmatrix}$ and define $\operatorname{Tr} A = \sum_{j=1}^{n} a_{jj}$. Then*

$$\begin{aligned} \operatorname{Tr} A &= \sum_{j=1}^{3} a_{jj} \\ &= 6 + 1 + 3 \\ &= 10 \end{aligned}$$

Next we will prove the cyclicty of trace using the definition of matrix multiplication.

Proposition 3.6. *The property that $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$ for all $A, B \in \mathcal{L}(V, V)$ is called the cyclicity of trace.*

$$\begin{aligned} \operatorname{Tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji} \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ij} a_{ij} \\ &= \sum_{j=1}^{n} (BA)_{jj} \\ &= \operatorname{Tr}(BA) \end{aligned}$$