REARRANGING INFINITE SERIES AND DOUBLE SUMMATION

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1. Introduction

The purpose of this paper is to look the rearrangement infinite series and the properties of the new resulting series. Specifically, we will decompose infinite series into positive and negative components and see that if we consider the supremum of finite sets over N, the order of the terms in the sequence doesn't matter. We'll then see that finite sums converge to finite values when both positive and negative portions are finite, that they converge to infinity or negative infinity when one sum is finite, and to any real value when both sums are infinite. We will conclude with a discussion of double summation and an examination of some of their properties.

2. Definition of Rearrangement

Definition 2.1. Let (b_n) be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \cdots .$$

We define the corresponding sequence of partial sums (s_m) by

$$s_m = b_1 + b_2 + b_3 + \cdots + b_m,$$

and say that the series $\sum_{n=1}^{\infty} b_n$ converges to B if the sequence (s_m) converges to B. In this case we write $\sum_{n=1}^{\infty} b_n = B$.

A rearrangement of a series is determined by using the same terms, but changing the order in which these terms occur.

Definition 2.2. Let $\sum_{n=1}^{\infty} b_n$ be a series. A series $\sum_{n=1}^{\infty} b_n$ is called a rearrangement of $\sum_{n=1}^{\infty} a_n$ if there exists a one-to-one, onto function such that $b_{f(n)} = b_n$ for all $n \in \mathbb{N}$

Example 2.3. *Consider the alternating harmonic series*

$$S_b = \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots .$$

Letting $S_a = \sum_{n=1}^{\infty} a_n$ *be a rearrangement of* $\sum_{n=1}^{\infty} b_n$ *such that there are two negative terms for each positive term gives the series*

$$\begin{aligned} S_a &= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} \\ &= (1 - \frac{1}{2}) - \frac{1}{4} + (\frac{1}{3} - \frac{1}{6}) - \frac{1}{8} + (\frac{1}{5} - \frac{1}{10}) \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} + \cdots \\ &= \frac{1}{2}[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots] \\ &= \frac{1}{2} S_b \end{aligned}$$

So we see that the sum of the rearranged series is equal to one half the original S. *In fact, it is well known that the alternating harmonic series converges to* $\ln 2$, *so this particular arrangement of terms is equal to* $\frac{1}{2}\ln 2$. *So we can see that the terms in an infinite series are not commutative.*

Let's look at a similar rearrangement. Let

$$S_c = \sum_{n=1}^{\infty} c_n = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} + \cdots$$

We can use our previous result to prove that $S_c = \frac{3}{2} S_b$

$$\frac{3}{2} S_b = S_b + \frac{1}{2} S_b$$

$$= [1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots] + \frac{1}{2}[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots]$$

$$= [1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots] + [\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} + \cdots]$$

$$= 1 - \frac{1}{2} + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{6} + \frac{1}{7} - \frac{1}{8} - \frac{1}{8} + \cdots$$

$$= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} + \cdots$$

$$= S_c$$

as desired.

3. Rearrangement of series with non-negative entries

Proposition 3.1. *If all the elements* $b_n \geq 0$*, then*

$$\sum_{n=1}^{\infty} b_n = \sup_{N \subset \mathbb{N}} \sum_{i \in N} b_i$$

where the supremum is taken over all subsets of $\mathbb{N} = \{1, 2, 3, \cdots\}$

Proof. By definition of a convergent series

$$\sum_{i=1}^{\infty} b_i = \lim_{n \to \infty} \sum_{i \in (1, \ldots, n)} b_i = \lim_{n \to \infty} B_n$$

We're given that $b_n \geq 0$, so B is increasing. Then

$$\lim B_n = \sup B_n$$

We want to show that $\sup\{B_n\} = \sup\{S\}$. Let $S = \sum_{i \in \mathbb{N}} b_i$ such that $N \in \mathbb{N}$ is finite. Then $\sup\{B_n\} \leq \sup\{S\}$ because $B_n \in S$. Now, for each $x \in S$, there exists a B_n with $B_n \geq x$ because $x = \sum_{i \in \mathbb{N}} b_i$ where N is finite. Let $n = \max N$ so $N \subset \{1, \ldots, n\}$. Then $B_n \geq x$ since $b_i \geq 0$ which implies $\sup\{B_n\} \geq \sup\{S\}$. Thus $\sup\{B_n\} = \sup\{S\}$ as desired. □

Corollary 3.2. *Any rearrangement of the series (3.1) converges to the same (possibly infinite) limit*

Proof. Let $\sum_{n=1}^{\infty} a_n$ be a rearrangement of $\sum_{n=1}^{\infty} b_n$. Then by definition, $\sigma : \mathbb{N} \to \mathbb{N}$ and $b_n = b_{\sigma(i)}$. i.e., a bijective function. By 3.2 we can say

$$\begin{aligned}
\sum_{i=1}^{\infty} b_{\sigma(i)} &= \sup_{N \subset \mathbb{N}} \sum_{i \in N} b_{\sigma(i)} \\
&= \sup_{N \subset \mathbb{N}} \sum_{i \in \sigma(N)} b_i \\
&= \sup_{N \subset \mathbb{N}} \sum_{i \in \mathbb{N}} b_i \\
&= \sum_{i=1}^{\infty} b_i
\end{aligned}$$

As desired. The first equality is given by 3.1. The second is because σ is bijective, meaning σ^{-1} exists, allowing us to take the supremum over $i \in \sigma(N)$. The third equality is allowed because rearranging finite sets does not change the sum and the final equality gives us our result by 3.1. □

4. Rearrangement of series with mixed signs

Definition 4.1. Given a real number b, define its *positive part* $b^+ = \max(b, 0)$ and define its *negative part* $b^- = \max(-b, 0)$.

Lemma 4.2. *Show $b = b^+ - b^-$.*

$$\begin{aligned} b &= b - 0 \\ &= \max(b, 0) - \max(-b, 0) \\ &= b^+ - b^- \end{aligned}$$

Definition 4.3. Given a series $\sum_{n=1}^{\infty} b_n$ define $S^+ = \sum_{n=1}^{\infty} b_n^+$ and $S^- = \sum_{n=1}^{\infty} b_n^-$.

Theorem 4.4. *If S^+ and S^- are both finite, then $\sum_{n=1}^{\infty} b_n = S^+ - S^-$.*

Proof. Let S^+ and S^- both be finite. By applying 4.2, series properties, and 4.3 we can easily see that

$$\begin{aligned} \sum_{i=1}^{\infty} b_n &= \sum_{i=1}^{\infty} (b_n^+ - b_n^-) \\ &= \sum_{i=1}^{\infty} b_n^+ - \sum_{i=1}^{\infty} b_n^- \\ &= S^+ - S^- \end{aligned}$$

□

Definition 4.5. A series $\sum_{n=1}^{\infty} b_n$ is said to be *absolutely convergent* if $\sum_{n=1}^{\infty} |b_n|$ is finite.

Proposition 4.6. *$\sum_{n=1}^{\infty} b_n$ is absolutely convergent if, and only if, S^+ and S^- are both finite*

Proof. ($\Longrightarrow$) $\sum_{n=1}^{\infty} b_n$ is absolutely convergent when S^+ and S^- are both finite.

We want to show that $\sum_{n=1}^{\infty} |b_n|$ is finite. Let S^+ and S^- both be finite. By 4.4,

$$S^+ - S^- = \sum_{n=1}^{\infty} b_n$$

Taking $|b_n|$ means $b_n \geq 0$ for all $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} |b_n| = \sup_{N \subset \mathbb{N}} \sum_{i \in N} b_i$$

which implies that $\sum_{n=1}^{\infty} |b_n|$ is finite as desired.

($\Longleftarrow$) S^+ and S^- are both finite when $\sum_{n=1}^{\infty} b_n$ is absolutely convergent.

Assume $\sum_{n=1}^{\infty} b_n$ converges absolutely. Then $\sum_{n=1}^{\infty} |b_n|$ is finite. Since $|b_n|$ means $b_n \geq 0$ for all n

$$\sum_{n=1}^{\infty} |b_n| = \sup_{N \subset \mathbb{N}} \sum_{i \in N} b_i$$

implies S exists and is finite. Now,

$$\begin{aligned} S &= S^+ - S^- \\ &= \sum_{i=1}^{\infty} b_i^+ - \sum_{i=1}^{\infty} b_i^- \end{aligned}$$

Again using 3.1, we observe that $b_n^+ \geq 0$ and $b_n^- \geq 0$ we see that

$$\begin{aligned} \sum_{i=1}^{\infty} b_i^+ &= \sup_{N \subset \mathbb{N}} \sum_{i \in N} b_i^+ \\ &= \sup S^+ \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{\infty} b_i^- &= \sup_{N \subset \mathbb{N}} \sum_{i \in N} b_i^- \\ &= \sup S^- \end{aligned}$$

implies S^+ and S^- are both finite as desired.

□

Corollary 4.7 (Corollary to Theorem 4.4). *Let $\sum_{n=1}^{\infty} b_n$ be a series in which S^+ or S^- is finite. Then any rearrangement of this series converges to the same limit.*

Proof. Assume if S^+ is infinite while S^- is finite, then $\sum_{n=1}^{\infty} b_n = \infty$; if S^+ is finite while S^- is infinite, then $\sum_{n=1}^{\infty} b_n = -\infty$ and let $\sum_{n=1}^{\infty} a_n$ be a rearrangement of $\sum_{n=1}^{\infty} b_n$.

$$\begin{aligned} \sum_{i=1}^{\infty} a_n &= \sum_{i=1}^{\infty} (a_n^+ - a_n^-) \\ &= \sum_{i=1}^{\infty} a_n^+ - \sum_{i=1}^{\infty} a_n^- \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} b_{f(n)}^{+} - \sum_{i=1}^{\infty} b_{f(n)}^{-} \\
&= \sum_{i=1}^{\infty} b_{n}^{+} - \sum_{i=1}^{\infty} b_{n}^{-} \\
&= S^{+} - S^{-}
\end{aligned}
$$

In the case that S^+ is infinite and S^- finite we get

$$
\begin{aligned}
\sum_{i=1}^{\infty} a_n &= \infty - S^- \\
&= \infty
\end{aligned}
$$

Because the sum or difference of any finite term with ∞ is ∞. Conversely, when S^+ is finite while S^- infinite we get

$$
\begin{aligned}
\sum_{i=1}^{\infty} a_n &= S^+ - \infty \\
&= -\infty
\end{aligned}
$$

as desired □

Example 4.8. *Compute S^+, S^- and $\sum_{n=1}^{\infty} b_n$ and verify the $S = S^+ - S^-$.*

Let $S = \sum b_n$ be an alternating geometric series of the form

$$\sum_{n=0}^{\infty} (-1)^n a r^n = a - ar + ar^2 - ar^3 + \cdots + ar^n - ar^{n+1} + \cdots$$

where a is the first term and r is the common ratio and $a \neq 0$. The lovely advantage of the geometric series is that we have a formula for the sums given that we know the values of a and r. The geometric series converges to $S = \frac{a}{1-r}$ when $a = 1$ and $S = a(\frac{1-r^n}{1-r})$ when $a \neq 1$ if and only if $0 < |r| < 1$.

Consider b_n when $a = 1$ and $r = \frac{1}{2}$

$$
\begin{aligned}
\sum_{n=1}^{\infty} b_n &= \sum_{n=0}^{\infty} (-1)^n (\frac{1}{2}) \\
&= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \cdots + (-\frac{1}{2})^n \\
&= \frac{1}{1 - \frac{1}{2}}
\end{aligned}
$$

$$= 2$$

Now,

$$S^+ = \sum_{i=1}^{\infty} b_n^+ = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots + \frac{1}{2^{2n}}$$

is a geometric series with $a = 1$ *and* $r = \frac{1}{4}$ *so*

$$\begin{aligned} S^+ &= \frac{1}{1 - \frac{1}{4}} \\ &= \frac{4}{3} \end{aligned}$$

and

$$S^- = \sum_{i=1}^{\infty} b_n^- = -\frac{1}{2} - \frac{1}{8} - \frac{1}{32} - \frac{1}{128} - \cdots - \frac{1}{2^{2n+1}}$$

is a geometric series with $a = -\frac{1}{2}$ *and* $r = \frac{1}{4}$ *which gives the sum*

$$\begin{aligned} S^- &= -\frac{1}{2}[\frac{1 - (\frac{1}{4})^n}{1 - \frac{1}{4}}] \\ &= -\frac{2}{3}[1 - \frac{2}{4^n}] \\ &= \frac{2}{(3)4^n} - \frac{2}{3} \end{aligned}$$

Where taking the limit of both sides clearly gives

$$lim_{n\to\infty} S^- = S^- = -\frac{2}{3}$$

Then

$$\begin{aligned} S &= S^+ - S^- \\ &= \frac{4}{3} - (-\frac{2}{3}) \\ &= 2 \end{aligned}$$

as desired

We've seen what happens in the case where both S^+ and S$^-arefiniteandthecasewhereeither$S$^+$ or S$^-isfinite.Whataboutwhenbothsumsarefinite?Wewillseethatinthispeculiarcase,wecanshowthatther$

Theorem 4.9. *Let* $\sum_{n=1}^{\infty} b_n$ *be a series such that* S^+ *and* S^- *are both infinite and* $\lim_{n\to\infty} b_n = 0$. *Given any number* $\beta \in \mathbb{R}$ *prove that there is a rearrangement of the series that converges to* β.

Proof. This proof is a constructive algorithm in which a process will be described and qualitatively and demonstrated in the following example. Let $\sum_{n=1}^{\infty} b_n$ be a series such that S^+

and S^- are both infinite and $\lim_{n\to\infty} b_n = 0$, and let $\sum_{n=1}^\infty a_n$ be a rearrangement of $\sum_{n=1}^\infty b_n$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ + \sum_{n=1}^{\infty} a_n^-$$

In words, this algorithm will add successive positive terms until until the partial sum is greater than β, after which it will add successive negative terms until the partial sum is below β. Since $S+$ and S^- are unbounded, we can repeat this process indefinitely until the series converges to any limit we choose, so long as the $(a_n) \to 0$. □

Example 4.10. *Show that the series $\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}$ satisfies $S^+ = \infty$ and $S^- = \infty$. Applying your algorithm in the proof of Theorem 4.9, give the first 15 terms of the rearrangement.*

let $\sum_{n=1}^\infty a_n$ be a rearrangement of $\sum_{n=1}^\infty b_n$, $\beta = 1$ and S^+, S^- be finite. Applying 4.9 *gives*

$$\sum_{n=1}^{\infty} a_n = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} - \frac{1}{6} + \frac{1}{11} + \frac{1}{13} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} + \frac{1}{15} - \frac{1}{14} + \cdots$$

which tends toward 1 *as desired.*

5. Double summations

Example 5.1. *The example in §2.1 discusses the dangerous ambiguity in defining $\sum_{i,j=1}^\infty a_{ij}$ as a double summation over two indexed variables where $a_{ij} = \frac{1}{2^{j-1}}$ if $j > i$, $a_{ij} = -1$ if $j = 1$, and $a_{ij} = 0$ if $j < i$. It turns out that*

$$\sum_{i,j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty}(\sum_{j=1}^{\infty} a_{ij}) = \sum_{i=1}^{\infty}(0) = 0$$

While

$$\sum_{i,j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}(\sum_{i=1}^{\infty} a_{ij}) = \sum_{j=1}^{\infty}(\frac{-1}{2^{j-1}}) = -2$$

demonstrating that order matters and we must develop a more rigorous understanding of double summations.

Theorem 5.2. *Let $\{a_{ij} : i, j \in \mathbb{N}\}$ be a doubly indexed array of real numbers. If*

$$S^+ = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}^+, \qquad S^- = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}^-$$

are both finite, then both $\sum_{i=1}^\infty \sum_{j=1}^\infty a_{ij}$ and $\sum_{j=1}^\infty \sum_{i=1}^\infty a_{ij}$ converge to the same value A. Moreover,

$$\lim_{n\to\infty} s_{nn} = A,$$

where $s_{nn} = \sum_{i=1}^n \sum_{j=1}^n a_{ij}$.

Proof. We want to show $S = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$. We are given that S^+ and S^- are finite. $S = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} = S^+ - S^-$ by 4.4. Then

$$\begin{aligned} A &= S^+ - S^- \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}^+ - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}^- \end{aligned}$$

Since S^+ is finite (and making a parallel argument for S^-), $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}^+ < \infty$ and $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}^+ < \infty$ so we can say

$$\begin{aligned} &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(a_{ij}^+ - a_{ij}^-) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \end{aligned}$$

As desired. Moreover, S^+ and S^- finite means $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$ is absolutely convergent so $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|$ is finite which implies $\lim_{n\to\infty} s_{nn} = A$

$\square$